

Comments on F -maximization and R-symmetry in 3D SCFTs

Vasilis Niarchos

Crete Centre for Theoretical Physics,
Department of Physics, University of Crete, 71003, Greece

niarchos@physics.uoc.gr

Abstract

We report preliminary results on the recently proposed F -maximization principle in 3D SCFTs. We compute numerically in the large- N limit the free energy on the three-sphere of an $\mathcal{N} = 2$ Chern-Simons-Matter theory with a single adjoint chiral superfield which is known to exhibit a pattern of accidental symmetries associated to chiral superfields that hit the unitarity bound and become free. We observe that the F -maximization principle produces a $U(1)$ R-symmetry consistent with previously obtained bounds but inconsistent with a postulated Seiberg-like duality. Potential modifications of the principle associated to the decoupling fields do not appear to be sufficient to account for the observed violations.

March, 2011

Contents

1. R-symmetry in three-dimensional $\mathcal{N} = 2$ SCFTs	1
2. $\hat{\mathbf{A}}$ theory	4
2.1. Definition and known facts	4
2.2. A brief note on moduli spaces	7
3. The partition function on S^3 and F -maximization in the large- N limit	7
3.1. The R-charge at $\lambda \sim \mathcal{O}(1)$	9
3.2. The R-charge at $\lambda \gg 1$	11
3.3. Comments on a potential discrepancy	12
4. Insights into the exact R-symmetry of the $\hat{\mathbf{A}}$ theory	13
5. Open problems	15

1. R-symmetry in three-dimensional $\mathcal{N} = 2$ SCFTs

Supersymmetric conformal field theories (SCFTs) with four real supersymmetries, namely $\mathcal{N} = 2$ supersymmetry in three dimensions, have a conserved $U(1)$ R-symmetry that sits in the same supermultiplet as the stress-energy tensor. In a general interacting SCFT this symmetry receives quantum corrections and the quantum numbers associated with it become non-trivial functions of the parameters that define the theory. The computation of the exact non-perturbative form of this symmetry is an important task with immediate implications, *e.g.* one can deduce from it the scaling dimensions of the chiral ring operators.

In recent years considerable progress has been achieved in this problem in the context of four-dimensional $\mathcal{N} = 1$ SCFTs. Ref. [1] showed that the exact superconformal R-symmetry is the one that maximizes a —the coefficient of the Euler density in the conformal anomaly.

More recently, evidence was presented [2] that the free energy of the Euclidean CFT on a three-sphere

$$F = -\log |Z_{S^3}| \tag{1.1}$$

plays a similar role in three-dimensional $\mathcal{N} = 2$ SCFTs. Putting the theory on a three-sphere in a manner that preserves supersymmetry requires the introduction of extra couplings between the matter fields and the curvature of S^3 . These couplings are determined by the choice of a trial R-symmetry. In this way, F is a function of the trial R-charges. It has been conjectured [2] that the exact $U(1)$ R-symmetry locally maximizes F .

So far, this proposal has passed a number of impressive tests [2, 3, 4, 5, 6, 7]. It reproduces known perturbative results and agrees with expectations from the AdS/CFT corre-

spondence. Currently, most of the analyzed examples refer to Chern-Simons-Matter quivers with $\mathcal{N} \geq 2$ supersymmetry and matter in the adjoint and bi-fundamental representation whose free energy scales in the large 't Hooft limit as $N^{3/2}$. A potential discrepancy with expectations from the AdS/CFT correspondence was briefly reported for theories with chiral bifundamental fields in [6].

More generally, it is natural to expect that the application of F -maximization will be subtle in cases with accidental global symmetries that are not visible in a weak coupling formulation of the theory. Accidental symmetries can arise at strong coupling to modify the result. Such situations are well known in four-dimensional $\mathcal{N} = 1$ SCFTs. For instance, in the case of accidental symmetries associated with fields hitting the unitarity bound it is known that the a -maximization principle should be modified appropriately to account for the decoupling fields [8]. The validity (and possible modifications) of the proposed F -maximization principle in three dimensions has not been considered in such situations so far.

Our goal in this paper is to test the F -maximization principle in an $\mathcal{N} = 2$ Chern-Simons-Matter (CSM) theory that is known to exhibit such strong coupling phenomena. The theory of interest is $U(N)$ $\mathcal{N} = 2$ Chern-Simons theory at level k coupled to a single chiral superfield X in the adjoint representation. It is believed that this theory is superconformal for all values of N, k [9]. Moreover, one can argue that the exact R-charge of the superfield X decreases towards zero as we make the theory more and more strongly coupled and that an increasing number of operators become free and decouple in this process. In fact, one can place specific non-perturbative constraints on how the exact R-charge decreases [10, 11]. The currently available information will be briefly reviewed in section 2. These constraints, which must be obeyed by the solution of any exact principle that determines the superconformal R-symmetry, like the proposed F -maximization, provide a novel way to check if the current formulation of F -maximization is valid, or in case it fails to detect how it fails and how it should be modified.

The partition function of $\mathcal{N} = 2$ SCFTs on S^3 reduces to a matrix integral [12, 2, 13] via localization [14]. In the case of the above single-adjoint CSM theory with gauge group $U(N)$ the matrix integral takes the form (up to an inconsequential overall factor)

$$Z_{S^3} \sim \int \left(\prod_{j=1}^N e^{\mathbf{i}\pi t_j^2 k} dt_j \right) \prod_{i < j}^N \sinh^2(\pi(t_i - t_j)) \prod_{i,j=1}^N e^{\ell(1-R+\mathbf{i}(t_i-t_j))} \quad (1.2)$$

where $\ell(z)$ is the function

$$\ell(z) = -z \log(1 - e^{2\pi \mathbf{i} z}) + \frac{\mathbf{i}}{2} \left[\pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi \mathbf{i} z}) \right] - \frac{\mathbf{i}\pi}{12} \quad (1.3)$$

R is the trial R-charge over which we are instructed to maximize the free energy (1.1). We integrate over the $N \times N$ matrix eigenvalues t_i .

We will focus on the following 't Hooft limit of the theory

$$N, k \rightarrow \infty, \quad \lambda = \frac{N}{k} = \text{fixed} . \quad (1.4)$$

In this limit the theory is parameterized by a single continuous parameter $\lambda \in [0, \infty)$.¹ Accordingly, the exact R-charge is a function of λ .

In lack of a tractable analytic method, we will compute the matrix integral (1.2) numerically in section 3 in the large- N limit (1.4) by solving a system of saddle point equations. Then we maximize the free energy to determine the exact R-symmetry. The result exhibits a function $R(\lambda)$ that decreases monotonically towards zero as we increase the coupling λ and remains in the vicinity of the lower bound derived in [11] (see (2.14)) without exhibiting any obvious signs of violation. A different type of potential violation is noted, where the numerically obtained behavior of the free energy appears to be inconsistent with the Seiberg-like duality postulated in [10]. A more detailed discussion of this aspect will appear elsewhere [15].

The effects of decoupling operators modify the F -maximization principle. However, in this particular case such effects are subleading in $1/N$ and do not appear to be capable of producing numerically significant corrections at finite 't Hooft coupling λ . It is currently unclear whether it is possible to find a modification of the F -maximization principle that resolves the tension with the Seiberg-like duality.

The F -maximization matrix integral solution indicates the possibility of a particular pattern of spontaneous supersymmetry breaking in the theories deformed by the superpotential interaction $W_{n+1} = \text{Tr } X^{n+1}$, $n = 1, 2, \dots$; a pattern where the point of supersymmetry breaking is the same point where the operator $\text{Tr } X^{n+1}$ becomes a free operator in the undeformed theory. If correct, this pattern would imply that the exact superconformal R-charge is an oscillating (presumably monotonic) function that trails closely the curve $\frac{1}{2(1+\lambda)}$. The current numerical results partially verify this intuition. Relevant comments appear in sec. 4.

In the final section 5 we conclude with a brief summary of the lessons of this work and a list of interesting open problems.

Note added. In the first version of this paper an erroneous violation of the bounds of Ref. [11] was reported due to a missing factor of 2 in the denominator of the second term in eq. (3.3). In the current version this error has been corrected and the numerical results updated.

¹We assume $k > 0$. The case of negative k can be obtained by a simple parity transformation.

New comments related to a potential disagreement with a postulated Seiberg-like duality have been added.

2. $\hat{\mathbf{A}}$ theory

2.1. Definition and known facts

We will focus on a particular class of three-dimensional $\mathcal{N} = 2$ SCFTs defined as $\mathcal{N} = 2$ Chern-Simons theory coupled to a single adjoint $\mathcal{N} = 2$ chiral superfield X . Following [11] we will refer to this theory (in the absence of superpotential deformations) as the $\hat{\mathbf{A}}$ theory.² The $\hat{\mathbf{A}}$ theory is characterized by two integers: the rank of the gauge group G (we will take $G = U(N)$ in this work), and the level of the Chern-Simons interaction k , which is also an integer. It is believed that this theory is exactly superconformal at the quantum level for any values of N, k [9].

In the large- N limit (1.4) there is a single continuous parameter, the 't Hooft coupling $\lambda = N/k$. The theory is weakly coupled when $\lambda \ll 1$, in which case we can treat it with standard perturbative techniques. In this regime the superconformal R-charge behaves as [9]

$$R(\lambda) = \frac{1}{2} - 2\lambda^2 + \mathcal{O}(\lambda^4) . \quad (2.1)$$

In this note we are interested in the exact non-perturbative version of (2.1).

There is no known holographic description of the $\hat{\mathbf{A}}$ theory. Ref. [9] explored the possibility of a holographic description in M-theory based on N M5-branes wrapping a special Lagrangian Lens space S^3/\mathbb{Z}_k . No AdS_4 solution was found for this system in supergravity which agrees with the expectation that α' corrections will be important in a type IIA dual string theory formulation of this theory. Hence, in this case we cannot invoke the AdS/CFT correspondence to gain information about the strong coupling behavior of the R-symmetry.

Instead, it is possible to obtain useful information about the exact R-symmetry by analyzing the properties of the theory under the superpotential deformations

$$W_{n+1} = \frac{g_{n+1}}{n+1} \text{Tr } X^{n+1} , \quad n = 1, 2, \dots . \quad (2.2)$$

As we review in a moment, there are regimes along the λ -line where these interactions are relevant and drive the theory to a new interacting IR fixed point. We will refer to the theory

²The more general class of $\hat{\mathbf{A}}$ theories defined in [11] includes also N_f pairs of (anti)fundamental chiral superfields Q^i, \tilde{Q}_i . These theories are three-dimensional cousins of adjoint-SQCD in four dimensions.

deformed by the chiral operator $\text{Tr } X^{n+1}$ as the \mathbf{A}_{n+1} theory. An argument based on a D-brane realization of this theory in string theory [10] shows that

(i) The superpotential deformation W_{n+1} lifts the supersymmetric vacuum for

$$N > nk \quad (\text{equivalently in the large-}N \text{ limit : } \lambda > n) . \quad (2.3)$$

(ii) The theory exhibits a Seiberg-like duality. The $U(N)$ theory at level k and superpotential deformation W_{n+1} is dual to the $U(nk - N)$ theory at the same level k and superpotential W_{n+1} . In the large- N 't Hooft limit the duality acts by taking

$$\lambda \rightarrow n - \lambda . \quad (2.4)$$

This picture has important implications for the R-symmetry in the undeformed theory $\hat{\mathbf{A}}$ [11]. Classically, *i.e.* at $\lambda \ll 1$, the chiral operators $\text{Tr } X^{n+1}$ ($n = 4, 5, \dots$) are all irrelevant and become more irrelevant the larger n is. The fact that there are values of $\lambda \in \mathbb{N}$ for which any operator $\text{Tr } X^{n+1}$ can lift the supersymmetric vacuum (no matter how large n is) implies that the exact R-charge decreases as we increase λ and eventually tends to zero at infinite λ .

More specifically, it implies that there is a sequence of critical couplings λ_{n+1}^* such that

$$0 = \lambda_2^* = \lambda_3^* = \lambda_4^* < \lambda_5^* < \dots < \lambda_n^* < \lambda_{n+1}^* < \dots \quad (2.5)$$

where each time one of the chiral operators $\text{Tr } X^{n+1}$ becomes marginal. By definition, λ_{n+1}^* is the value of the 't Hooft coupling where the operator $\text{Tr } X^{n+1}$ has scaling dimension

$$\Delta(\text{Tr } X^{n+1}) = (n+1)R(\lambda_{n+1}^*) = 2 \quad \Leftrightarrow \quad R(\lambda_{n+1}^*) = \frac{2}{n+1} . \quad (2.6)$$

Clearly, the generic operator $\text{Tr } X^{n+1}$ must become marginal before it becomes capable to lift the supersymmetric vacuum at $\lambda_{n+1}^{\text{SUSY}} = n$. This implies

$$\lambda_{n+1}^* < n \quad (2.7)$$

and

$$\Delta(\text{Tr } X^{n+1})|_{\lambda=n} = (n+1)R(n) < 2 \quad \Leftrightarrow \quad R(n) < \frac{2}{n+1} . \quad (2.8)$$

A more strict upper bound on λ_{n+1}^* can be deduced from the Seiberg-like duality (2.4). Requiring the existence of a finite range of λ -values within which the deforming operator $\text{Tr } X^{n+1}$ is relevant both in the $U(N)$ theory and its $U(nk - N)$ dual implies

$$\lambda_{n+1}^* < n - \lambda_{n+1}^* \quad \Leftrightarrow \quad \lambda_{n+1}^* < \frac{n}{2} . \quad (2.9)$$

$n - \lambda_{n+1}^*$ is the point where $\text{Tr } X^{n+1}$ becomes marginal in the dual theory. The interval $[\lambda_{n+1}^*, n - \lambda_{n+1}^*]$ plays in the \mathbf{A}_{n+1} theory the analog of the standard conformal window in 4d SQCD. The point $\lambda = \frac{n}{2}$ is a self-dual point for Seiberg duality in the \mathbf{A}_{n+1} theory.

As we increase λ beyond some $\lambda_{n'+1}^*$ we reach the critical coupling $\lambda_{n'+1}^*$ ($n' > n$) of another operator $\text{Tr } X^{n'+1}$. It so happens that there is an integer n' for which $\text{Tr } X^{n'+1}$ is marginal and simultaneously $\text{Tr } X^{n+1}$ hits the unitarity bound and becomes free. This occurs precisely when

$$\Delta(\text{Tr } X^{n+1}) = (n+1)R(\lambda_{n'+1}^*) = \frac{2(n+1)}{n'+1} = 2 \quad \Leftrightarrow \quad n' = 4n + 3. \quad (2.10)$$

Once we reach $\lambda_{4(n+1)}^*$, where the operator $\text{Tr } X^{n+1}$ becomes free, we cannot use it any longer to deform the $\hat{\mathbf{A}}$ theory without destabilizing the supersymmetric vacuum. Hence, the spontaneous supersymmetry breaking point $\lambda_{n+1}^{\text{SUSY}} = n$ of the \mathbf{A}_{n+1} theory cannot be greater than $\lambda_{4(n+1)}^*$. That implies a further pair of inequalities [11]

$$n \leq \lambda_{4(n+1)}^*, \quad (2.11)$$

and

$$\Delta(\text{Tr } X^{n+1})|_{\lambda=n} = (n+1)R(n) \geq \frac{1}{2} \quad \Leftrightarrow \quad R(n) \geq \frac{1}{2(n+1)}. \quad (2.12)$$

Combining the inequalities (2.9), (2.8), (2.11), (2.12) we find

$$\left\lceil \frac{n-3}{4} \right\rceil \leq \lambda_{n+1}^* < \frac{n}{2}, \quad (2.13)$$

and

$$\frac{1}{2(n+1)} \leq R(n) < \frac{2}{n+1}, \quad n = 1, 2, \dots \quad (2.14)$$

Assuming $R(\lambda)$ is a monotonically decreasing function,³ the improved upper bound on λ_{n+1}^* in (2.13) further implies

$$R(\lambda) < \frac{2}{2\lambda+1}, \quad \text{for } \lambda = \frac{n}{2}, \quad n = 1, 2, \dots \quad (2.15)$$

In the next section we will test whether F -maximization in its current form obeys these inequalities.

³The intuition that gauge interactions work to decrease the R-charge with increasing λ makes this assumption plausible. However, it is far from obvious that this is a correct statement in the exact theory. We will see that the R-charge derived from F -maximization satisfies this property of monotonicity. We stress that the validity of the inequalities (2.13), (2.14) does not rely on this assumption.

To summarize, in the $\hat{\mathbf{A}}$ theory we encounter the following situation. At weak coupling the operator $\text{Tr } X$ is free and decoupled but all the other chiral ring operators $\text{Tr } X^{n+1}$ ($n > 0$) are interacting. As we increase λ more and more of the operators from the chiral ring are decommissioned. For any n there is always a value of λ above which the scaling dimension of the operator $\text{Tr } X^{n+1}$ takes the free field value $\frac{1}{2}$. According to standard lore, the operator becomes a free field at that point and decouples from the rest of the theory. Hence, with increasing λ more and more of the bottom part of the chiral ring decouples.

2.2. A brief note on moduli spaces

Before moving to the computation of the free energy on the three-sphere let us interject a comment on the deformation W_{n+1} that gives rise to the theories \mathbf{A}_{n+1} .

By definition, the operator $\text{Tr } X^{n+1}$ is marginal at $\lambda = \lambda_{n+1}^*$. We would like to ask: is the superpotential deformation W_{n+1} an exactly marginal deformation at λ_{n+1}^* ?

The technology of Ref. [16] allows us to give a definite answer to this question. The superpotential deformation (2.2) breaks the global $U(1)_X$ symmetry that rotates the superfield X and gives a non-vanishing D^a (in the language of [16]). Hence, g_{n+1} at $\lambda = \lambda_{n+1}^*$ is a marginally-irrelevant coupling. This is similar to what happens with classically marginal superpotential deformations in Wess-Zumino models.

As we increase λ above λ_{n+1}^* , the superpotential deformation (2.2) becomes relevant and there is a flow towards a fixed point (the superconformal field theory we denote by \mathbf{A}_{n+1}). At this point the superpotential coupling g_{n+1} becomes a function of λ

$$g = g_{n+1}(\lambda) . \tag{2.16}$$

This can be shown explicitly in conformal perturbation theory when $\lambda - \lambda_{n+1}^* \ll 1$ (see *e.g.* [16]). Intuitively, an IR fixed point arises from a balancing of two counteracting sources: the gauge interactions that work to decrease the R-charges and the superpotential interactions that work to increase them.

3. The partition function on S^3 and F -maximization in the large- N limit

We proceed to compute the matrix integral (1.2) in the large- N limit (1.4), implement the F -maximization principle and determine R as a function of λ . Expressed as a function

of N, λ, R the partition function Z_{S^3} reads

$$Z_{S^3} = \int \left(\prod_{j=1}^N e^{\frac{i\pi N}{\lambda} t_j^2} dt_j \right) \prod_{i < j}^N \sinh^2(\pi(t_i - t_j)) \prod_{i,j=1}^N e^{\ell(1-R+i(t_i-t_j))} = e^{-\mathcal{F}(N,\lambda,R)} . \quad (3.1)$$

The free energy that we want to maximize with respect to R is

$$F = \frac{1}{2}(\mathcal{F} + \bar{\mathcal{F}}) . \quad (3.2)$$

In the large- N limit the main contribution to Z_{S^3} comes from saddle point configurations that obey the system of algebraic equations

$$\mathcal{I}_i \equiv \frac{i}{\lambda} t_i + \frac{1}{N} \sum_{j \neq i}^N \left[\coth(\pi t_{ij}) - \frac{(1-R) \sinh(2\pi t_{ij}) + t_{ij} \sin(2\pi R)}{\cosh(2\pi t_{ij}) - \cos(2\pi R)} \right] = 0 , \quad i = 1, 2, \dots, N \quad (3.3)$$

where we have defined

$$t_{ij} = t_i - t_j . \quad (3.4)$$

In general, the N t_i 's that solve these equations are complex numbers. In this case, they are \mathbb{C} -valued functions of the parameters R, λ .

At a saddle point configuration

$$-\mathcal{F}(\lambda, N) = \sum_{i=1}^N \frac{i\pi N}{\lambda} t_i^2 + \sum_{i < j}^N \log \sinh^2(\pi t_{ij}) + \sum_{i,j=1}^N \ell(1-R+i t_{ij}) . \quad (3.5)$$

We are not aware of an efficient analytic method of solution of these equations in this particular case, so we will proceed with a more elementary numerical approach. As pointed out in [3] it is convenient to view such equations as equations describing the equilibrium configuration of N point particles whose coordinates are given by the complex numbers t_i . The equilibrium configuration can be found by introducing a fictitious time coordinate τ and considering the dynamical evolution described by the differential equation

$$a \frac{dt_i}{d\tau} = \mathcal{I}_i . \quad (3.6)$$

By suitably choosing the constant a we can arrange for solutions that converge very quickly to an equilibrium configuration described by the system (3.3).

We implemented this approach on the computer (with the use of MATHEMATICA) for a wide range of λ and R values. As a typical value of large N we used $N = 100$. An appropriate choice of a is $a = -i$.

A typical saddle point configuration appears in Fig. 1. The configuration is symmetric under the transformation $t_i \rightarrow -t_i$ as is evident from the equations (3.3). The specific

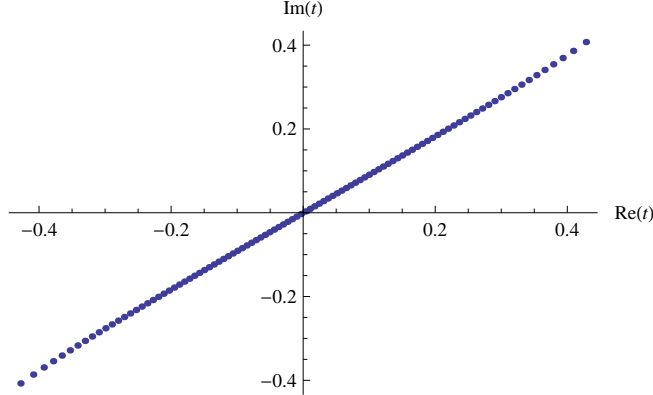


Figure 1: A typical distribution of the eigenvalues t_i ($i = 1, \dots, N$) in the complex plane. This particular plot has been obtained for $N = 100$, $\lambda = 1$ and $R = 0.28$.

arrangement of the eigenvalues in the complex plane depends on the values of λ and R . However, in general, we observe that the eigenvalues are oriented along a line of approximately 45 degrees and that the size of their domain increases with increasing λ (at fixed N and R).

Maximizing the free energy (3.2) with respect to R we determine the R-charge as a function of λ . The result of this calculation is plotted in Figs. 2, 3. Let us discuss separately the regimes with λ of order one and $\lambda \gg 1$.

3.1. The R-charge at $\lambda \sim \mathcal{O}(1)$

Fig. 2 zooms into the region of interest. This region includes the perturbative regime of $\lambda \ll 1$, but also a regime of strong coupling at λ of order 1 (we present data up to $\lambda = 10$). In this regime we find that the free energy F scales with N as $\mathcal{O}(N^2)$.

At very weak coupling the curve follows very closely the result of the perturbative calculation (2.1). The successful matching of the perturbative result with the result obtained from F -maximization was noticed already in [2] for gauge group $SU(2)$ and more recently in [7] for the general $SU(N)$ case. In this paper we are considering the case of $U(N)$ gauge group. It is not difficult to show, using a trick in [7], that the $SU(N)$ and $U(N)$ versions of the matrix integral (3.1) are simply related by the equation

$$Z_{S^3}[SU(N)] = \frac{1}{\sqrt{i\lambda}} e^{-\ell(1-R)} Z_{S^3}[U(N)] . \quad (3.7)$$

Hence, to leading order in the planar limit, F -maximization leads to the perturbative result (2.1) in the $U(N)$ case as well in agreement with expectations.

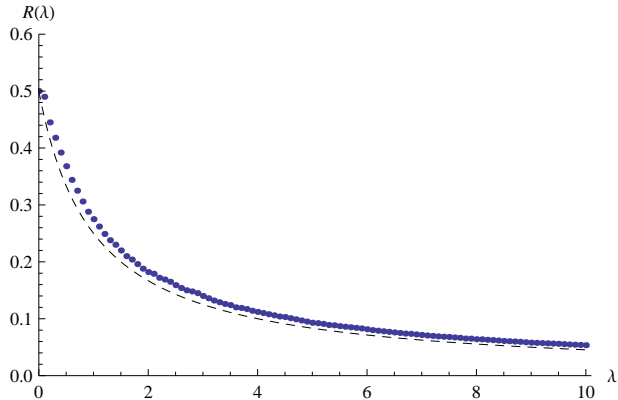


Figure 2: The numerically computed R -charge curve in the regime of $\lambda \sim \mathcal{O}(1)$. The dashed curve represents the lower bounding function $\frac{1}{2(1+\lambda)}$.

λ	1	2	3	4	5	6	7	8	9	10
R_{num}	0.275	0.182	0.14	0.112	0.093	0.0815	0.071	0.064	0.058	0.0535
R_{bound}	0.25	0.1667	0.125	0.1	0.083	0.0714	0.0625	0.0556	0.05	0.0455

Table 1: Numerically determined values of R at $\lambda = 1, 2, \dots, 10$ and the corresponding lower bounding values.

Away from the perturbative regime we observe the R -charge decreasing monotonically.⁴ It remains well below the two upper bounds and close, but above, the lower bound set by the curve $\frac{1}{2(1+\lambda)}$. Recall that this curve places a lower bound on R only when λ is an integer. A list of the numerically determined values of R at $\lambda = 1, 2, \dots, 10$ and the corresponding lower bounding values appears in Table 1. As we increase $\lambda = n \in \mathbb{N}$ the difference $R_{num}(n) - R_{bound}(n)$ decreases but remains positive respecting the bound (2.14). By analyzing the N -dependence of the numerical results we find that the typical error is of the order of a few percent. The difference $R_{num} - R_{bound}$ in Table 1 is of the order of 10%.

The first chiral operator that saturates the unitarity bound is $\text{Tr } X$. As is evident already from the perturbative result (2.1) $\text{Tr } X$ is a free operator at any value of λ .

Non-perturbatively we observe that the second operator that hits the unitarity bound is $\text{Tr } X^2$. That occurs very close to $\lambda = 1$, equivalently

$$\Delta \left(\text{Tr } X^2 \right) \Big|_{\lambda=1} - \frac{1}{2} = 2R(1) - \frac{1}{2} \sim 0.025 . \quad (3.8)$$

As we increase λ more and more single-trace operators decouple sequentially. As we reviewed

⁴We have checked (for the saddle point solutions reported in this paper) that F has a single extremum in the regime of this subsection.

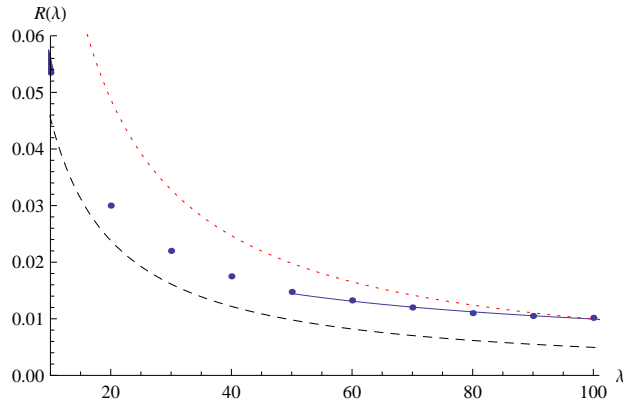


Figure 3: Numerical results in the regime of very strong 't Hooft coupling. The black and red curves represent the lower bounding function $\frac{1}{2(1+\lambda)}$ and the upper bounding function $\frac{2}{2\lambda+1}$ respectively.

in section 2, $\lambda = n \in \mathbb{Z}$ is a special coupling in the \mathbf{A}_{n+1} theory. For example, the \mathbf{A}_2 theory is the mass-deformed $\hat{\mathbf{A}}$ theory. In that case, in the far infrared the deformed theory flows to the $\mathcal{N} = 2$ Chern-Simons theory, a topological theory known to exhibit spontaneous supersymmetry breaking at $\lambda > 1$ [17]. Here we observe numerically that the deforming operators $\text{Tr } X^{n+1}$ come very close to becoming free as we approach the supersymmetry breaking point of the \mathbf{A}_{n+1} theory. We will return to this point in section 4.

3.2. The R-charge at $\lambda \gg 1$

As we move into the regime of stronger and stronger 't Hooft coupling the numerically determined R-charge curve continues to behave in a monotonically decreasing fashion approaching zero. The numerical computation becomes slower and the N -dependence increases. In addition, order- $[\lambda]$ operators are decoupled. Accordingly, the effect of this decoupling is expected to increase as we increase λ .

As an indicative illustration of the solutions of F -maximization in this regime we exhibit in Fig. 3 numerical results for $N = 100$ and λ up to N . A fit of the data for λ above 50 provides the following estimate for the asymptotic behavior

$$R(\lambda) \sim 0.119 \lambda^{-0.538} . \quad (3.9)$$

The numerical curve appears to cross the upper bounding curve $\frac{2}{2\lambda+1}$ in the vicinity of $\lambda \sim 100$. In this paper we will refrain from drawing specific conclusions from this result and postpone a more detailed exploration of this regime to a future publication.

At the same time, we observe that the free energy continues to scale like N^2 . In the language of [6] this scaling is due to the non-cancellation of long-range forces on the eigenvalues in the saddle point approximation.

3.3. *Comments on a potential discrepancy*

The above application of the F -maximization recipe does not produce any clear violations of the bounds of section 2 in the regime of subsection 3.1. In principle, when some operators become free and decouple from the rest of theory new accidental symmetries occur that can mix with the R-symmetry. In such cases, the exact R-symmetry that we want to determine does not refer to the decoupled operators anymore and any extremization principle should properly take this fact into account by ‘subtracting’ the free operators. Similar modifications of the a -maximization principle in four-dimensional SCFTs have been discussed in [8]. In that case, one is instructed to maximize a modified a -function where the ‘t Hooft anomalies associated to the decoupled fields have been subtracted. Clearly, a similar ‘subtraction’ recipe must be implemented in the case of F -maximization.

In the regime of subsection 3.1 only a finite (order-one) number of single-trace operators decouples. Hence, in the absence of other effects, such ‘subtraction’ modifications are expected to have negligible effects to the free energy (an order- N^2 quantity) and F -maximization should go through unobstructed. From this point of view, it is good news that no violations of the bounds of section 2 are observed in this regime. On a related note, it is interesting to compare the current situation with the corresponding situation in adjoint-SQCD in four dimensions [8]. In that case, the fields whose decoupling produces a sizable effect in the a -maximization procedure in the large- N limit are mesonic fields whose number is order- N^2 , a number scaling similarly to the free energy.

The non-perturbative bounds reviewed in section 2 is only part of the picture expected to describe the $\hat{\mathbf{A}}$ and \mathbf{A}_{n+1} theories. Another important part of the story is the Seiberg-like duality in the \mathbf{A}_{n+1} theories proposed in [10] (where also several checks of the duality were performed). In the large- N limit this duality maps the \mathbf{A}_{n+1} theory at coupling λ to the \mathbf{A}_{n+1} theory at coupling $n - \lambda$. This duality has several non-trivial consequences. One of them is the prediction that inside the ‘conformal window’ $(\lambda_{n+1}^*, n - \lambda_{n+1}^*)$ the free energies of the dual theories at λ and $n - \lambda$ should match. Since localization is blind to the superpotential interactions, we can compute these free energies with no extra effort simply by setting R in the previous matrix integral computation to the value dictated by the requirement that the superpotential interaction is marginal (in the case of the \mathbf{A}_{n+1} theory that is $R = \frac{2}{n+1}$).

Then, the Seiberg-like duality predicts the following set of equations

$$\mathcal{F}\left(\frac{2}{n+1}, \lambda\right) = \mathcal{F}\left(\frac{2}{n+1}, n - \lambda\right) \quad , \quad \lambda \in (\lambda_{n+1}^*, n - \lambda_{n+1}^*) \quad , \quad n = 3, 4, \dots \quad (3.10)$$

which allow us to probe the validity of the matrix integral (1.2) (and F -maximization) over a wider range of R and λ values.

Preliminary results based on the solutions of the saddle point equations presented above show that the oscillatory behavior implied by the equations (3.10) is not observed. Assuming the validity of the duality, this implies the presence of additional effects above $\lambda \sim 1$ which have to be taken properly into account in order to make sense of F -maximization in that regime. A detailed discussion of these issues will be presented in [15].

4. Insights into the exact R-symmetry of the $\hat{\mathbf{A}}$ theory

In anticipation of a modified F -maximization principle that gives the exact R-symmetry at arbitrary 't Hooft coupling we would like to offer in this section a few tentative comments on a possible result. We warn the reader that some of the statements that follow are speculative and we currently have no independent conclusive means to check whether this scenario is actually realized by the exact R-symmetry.

In subsection 3.1 we observed that the R-charge obtained from F -maximization remains in the vicinity of the curve $\frac{1}{2(1+\lambda)}$. The current formulation of F -maximization seems to be trustable at least within the regime $\lambda \in [0, 1]$, where it reproduces the perturbative result correctly and there are no obvious discrepancies with known or expected facts. We would like to suggest that the proximity of the R-charge curve to the bounding curve $\frac{1}{2(1+\lambda)}$ remains true at arbitrary values of λ .

Part of our intuition about this property comes from a qualitatively similar theory in four dimensions, $SU(N_c)$ SQCD with N_f fundamental/antifundamental superfield pairs and a single adjoint superfield. This theory was analyzed with a -maximization techniques in [8]. In the large- N limit, there is a single parameter that controls the dynamics, the ratio $x = \frac{N_c}{N_f}$. This theory, which is asymptotically free for $x > \frac{1}{2}$, is believed to flow in the IR to a non-trivial fixed point. When deformed by the superpotential $W_{n+1} \sim \text{Tr } X^{n+1}$ one finds that there is a supersymmetric vacuum only when $x < n$. The exact R-symmetry can be determined with a -maximization, properly defined to account for decoupling fields. In particular, the R-charge of the adjoint superfield is found to be monotonically decreasing

function of x that approaches zero at large x with the asymptotics

$$R(x) \sim \frac{4 - \sqrt{3}}{3x} . \quad (4.1)$$

Hence, at $x = n \in \mathbb{N}$ the corresponding asymptotics of the scaling dimension of the generic single trace operator $\text{Tr } X^{n+1}$ obeys the relation

$$\Delta \left(\text{Tr } X^{n+1} \right) \Big|_{x=n} - 1 = \frac{3}{2}(n+1)R(n) - 1 \sim 1 - \frac{\sqrt{3}}{2} \sim 0.13 . \quad (4.2)$$

The operator $\text{Tr } X^{n+1}$ comes close to becoming a free field at the point where the supersymmetric vacuum is lifted in the deformed theory and the R-charge curve trails the lower bound curve $\frac{2}{3(1+\lambda)}$ that follows from the inequality

$$\frac{2}{3(1+n)} < R(n) , \quad n \in \mathbb{N} . \quad (4.3)$$

Of course, there are important differences between the three-dimensional CSM theory that we are discussing here and the above four-dimensional adjoint-SQCD theory. For example, the latter has necessarily a non-vanishing number of fundamentals. Despite this fact, we propose that it is not unreasonable to anticipate some qualitative similarities (indeed, we already observe several non-trivial similarities, *e.g.* similarities in the supersymmetry breaking pattern).

Furthermore, given the small numerical value on the rhs of eq. (3.8), we would like to take the above picture one step further and suggest the possibility of the following property of the exact R-symmetry⁵

$$\Delta \left(\text{Tr } X^{n+1} \right) \Big|_{\lambda=n} = \frac{1}{2} \quad \Leftrightarrow \quad R(n) = \frac{1}{2(1+n)} . \quad (4.4)$$

If correct, this property implies that the deformation $W_{n+1} \sim \text{Tr } X^{n+1}$ yields an IR fixed point with a supersymmetric vacuum all the way up to the point where the deforming operator becomes free. Since we have a theory of a single chiral superfield in this case, this property may not be unnatural. It seems less likely to be exact in situations that involve the dynamics of extra fields, for example extra fields in the fundamental representation.

⁵The numerical results of the previous section suggest that these equalities are approximately but not exactly correct (at least for small enough λ). Since the deviation from (4.4) is of a few percent, and thus comparable with the expected numerical error, we will not use the numerical results here to make a definite conclusive statement about the fate of the equalities (4.4) and will instead proceed to explore their implications.

Assuming the validity of (4.4) as a working hypothesis we arrive at the following picture for the λ -dependence of the exact R-symmetry of the $\hat{\mathbf{A}}$ theory.

In principle, the exact R-charge curve can exhibit one of the following three different types of dependence on λ :

(i) $R(\lambda)$ oscillates indefinitely in the vicinity of the function

$$f(\lambda) = \frac{1}{2(1+\lambda)} \quad (4.5)$$

passing through the points (4.4) at $\lambda = n \in \mathbb{N}$.

(ii) $R(\lambda)$ coincides with the function $f(\lambda)$.

(iii) $R(\lambda)$ oscillates in a finite interval of λ and coincides with the function $f(\lambda)$ in its complement.

The only feasible possibility is possibility (i). Possibility (ii) is excluded immediately by the perturbative result (2.1). Possibility (iii) is excluded by the invariance of R under the transformation $k \rightarrow -k$, or equivalently $\lambda \rightarrow -\lambda$. This transformation can also be seen as a parity transformation. Although this is not a symmetry of the theory, it is a symmetry of the spectrum and hence a symmetry of the R-charge.

Hence, under the assumption (4.4), we conclude that the R-charge is an oscillating (presumably monotonic) function of λ with the following asymptotics at strong coupling

$$R(\lambda) \sim \frac{1}{2\lambda} . \quad (4.6)$$

It will be interesting to verify how close to reality the above picture is.

5. Open problems

We argued that the application of F -maximization is a subtle exercise in a three-dimensional SCFT with accidental symmetries associated with fields that reach the unitarity bound and decouple. In a regime where the effects of the decoupling fields appear to be negligible we found evidence that F -maximization respects the non-perturbative bounds of [11] but fails to reproduce results consistent with the Seiberg-like duality of [10]. Since our arguments were based solely on a numerical computation it would be useful to substantiate them further with additional analytic evidence.

The main remaining tasks are: (i) to determine conclusively if and how the current application of F -maximization fails, and resolve the puzzles that have emerged, and (ii) if modifications are needed to determine them and obtain results consistent with known facts. If (ii) can be successfully implemented, it is interesting to verify or disprove whether the exact R -symmetry behaves in the manner anticipated in section 4. Our ultimate hope is to obtain lessons that are applicable beyond the specific theory that was discussed in this paper.

For example, it would be very interesting to examine the corresponding properties of the more general class of $\hat{\mathbf{A}}$ theories in [11] that include N_f additional pairs of fundamental/antifundamental superfields Q^i, \tilde{Q}_i . In these theories there are two unknown R -charges, $R(Q)$ and $R(X)$, which are non-trivial functions of the parameters $x = \frac{N_f}{N}$ and $\lambda = \frac{N}{k}$ (in the large- N , N_f limits). It is known [11] that $R(X)$ decreases at strong coupling towards a non-zero value $R(X)_{\text{lim}} > \frac{1}{2([x]+2)}$. No particular information is currently available for $R(Q)$ beyond the perturbative regime. It is possible that as we increase the 't Hooft coupling some meson-like operators hit the unitarity bound and decouple. In that sense, this particular example may prove more appropriate in studying modifications of the F -maximization principle associated to decoupling fields.

Finally, a related issue has to do with the postulated F -theorem in [6], which states that the free energy F on the three-sphere (1.1) decreases along RG flows and plays the role of a c -function in three dimensions. Potential modifications of the F -maximization principle may have direct implications to the formulation of an F -theorem as well.

Acknowledgements

I would like to thank Elias Kiritsis and Dario Martelli for helpful comments and discussions. I am also grateful to David Kutasov for helpful discussions and correspondence, Daniel Jafferis for explaining a point in Ref. [6] and Takeshi Morita for the collaboration in a related project. This work was partially supported by the European Union grants FP7-REGPOT-2008-1-CreteHEPCosmo-228644 and PERG07-GA-2010-268246.

References

- [1] K. A. Intriligator and B. Wecht, “The exact superconformal R -symmetry maximizes a,” Nucl. Phys. B **667**, 183 (2003) [arXiv:hep-th/0304128].

- [2] D. L. Jafferis, “The Exact Superconformal R-Symmetry Extremizes Z ,” arXiv:1012.3210 [hep-th].
- [3] C. P. Herzog, I. R. Klebanov, S. S. Pufu, T. Tesileanu, “Multi-Matrix Models and Tri-Sasaki Einstein Spaces,” Phys. Rev. **D83**, 046001 (2011). [arXiv:1011.5487 [hep-th]].
- [4] D. Martelli, J. Sparks, “The large N limit of quiver matrix models and Sasaki-Einstein manifolds,” [arXiv:1102.5289 [hep-th]].
- [5] S. Cheon, H. Kim, N. Kim, “Calculating the partition function of $N=2$ Gauge theories on S^3 and AdS/CFT correspondence,” [arXiv:1102.5565 [hep-th]].
- [6] D. L. Jafferis, I. R. Klebanov, S. S. Pufu, B. R. Safdi, “Towards the F-Theorem: $N=2$ Field Theories on the Three-Sphere,” [arXiv:1103.1181 [hep-th]].
- [7] A. Amariti, “On the exact R charge for $N=2$ CS theories,” [arXiv:1103.1618 [hep-th]].
- [8] D. Kutasov, A. Parnachev and D. A. Sahakyan, “Central charges and $U(1)_R$ symmetries in $N = 1$ super Yang-Mills,” JHEP **0311**, 013 (2003) [arXiv:hep-th/0308071].
- [9] D. Gaiotto and X. Yin, “Notes on superconformal Chern-Simons-matter theories,” JHEP **0708**, 056 (2007) [arXiv:0704.3740 [hep-th]].
- [10] V. Niarchos, “Seiberg Duality in Chern-Simons Theories with Fundamental and Adjoint Matter,” JHEP **0811**, 001 (2008) [arXiv:0808.2771 [hep-th]].
- [11] V. Niarchos, “R-charges, Chiral Rings and RG Flows in Supersymmetric Chern-Simons-Matter Theories,” JHEP **0905**, 054 (2009) [arXiv:0903.0435 [hep-th]].
- [12] A. Kapustin, B. Willett and I. Yaakov, “Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter,” JHEP **1003**, 089 (2010) [arXiv:0909.4559 [hep-th]].
- [13] N. Hama, K. Hosomichi, S. Lee, “Notes on SUSY Gauge Theories on Three-Sphere,” JHEP **1103**, 127 (2011). [arXiv:1012.3512 [hep-th]].
- [14] V. Pestun, “Localization of gauge theory on a four-sphere and supersymmetric Wilson loops,” arXiv:0712.2824 [hep-th].
- [15] T. Morita and V. Niarchos, work in progress.
- [16] D. Green, Z. Komargodski, N. Seiberg, Y. Tachikawa and B. Wecht, “Exactly Marginal Deformations and Global Symmetries,” JHEP **1006**, 106 (2010) [arXiv:1005.3546 [hep-th]].
- [17] E. Witten, “Supersymmetric index of three-dimensional gauge theory,” In *Shifman, M.A. (ed.): The many faces of the superworld* 156-184. [hep-th/9903005].